# A NEW SUBSOIL MODEL FOR THE ANALYSIS SOIL-STRUCTURE INTERACTION

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**Summary:**
Three-dimensional description of building structure taking into consideration soil-structure interaction is a very complex problem and solution of this problem is often obtained by using finite element method. However, this method takes a significant amount of computational time and memory. Therefore, an efficient semi-analytical description procedure of subsoil, that could provide accurate results with significantly reduced computational time is proposed in this study for the analysis soil-structure-interaction. The model allows the description to be made in three dimensional scheme and takes into account the complicated characteristic of subsoil strata. Because of the small number of unknowns the computation can be carried out easily on commonly used hardware of PC class. The examples prove the efficiency and the computing possibilities of the model.

**Keywords:** subsoil model, soil-structure interaction, finite element method, three dimensional analysis

## Introduction

Soil-structure interaction is a very complex problem and the solution of this problem needs not only the suitable choice of structure model (Chyzy et al. 2006, Choi et al. 1996, 1998, 1999) but also the suitable choice of subsoil model. Much work describing the structure and subsoil contribution indicates wide application of finite element method (FEM) in the domain (Wang and Cheung 2001, Kundu et al. 1991, Miedzialowski 1996, Viladkar et al. 1995). Three-dimensional classical finite element analysis of building structure-subsoil system incorporates substantial disadvantages as a result of high time-consuming computations (high number of degrees of freedom) and extended data processing. Elaboration of the subsoil model significantly delimiting number degree of system freedom seems to be sensible.

This paper deals with three dimensional model of subsoil via finite element method, applicable in scientific research and engineering practice in the static analysis of the spatial building structures. The prismatic elements separated from soil space under the footing or strip foundation interconnected through spatial coupling elements are the roots of the model. The displacement far field around the foundation is simulated by using infinite elements.

Presented model of subsoil is compatible with many models (based on FEM) of the structure but especially by semi-analytical model (MQDES) of building structure presented by (Chyzy et al. 2006).

## The soil-structure interaction model

Fig.1. shows the soil-structure interaction model which takes into account the building structure model, contact area and subsoil model.

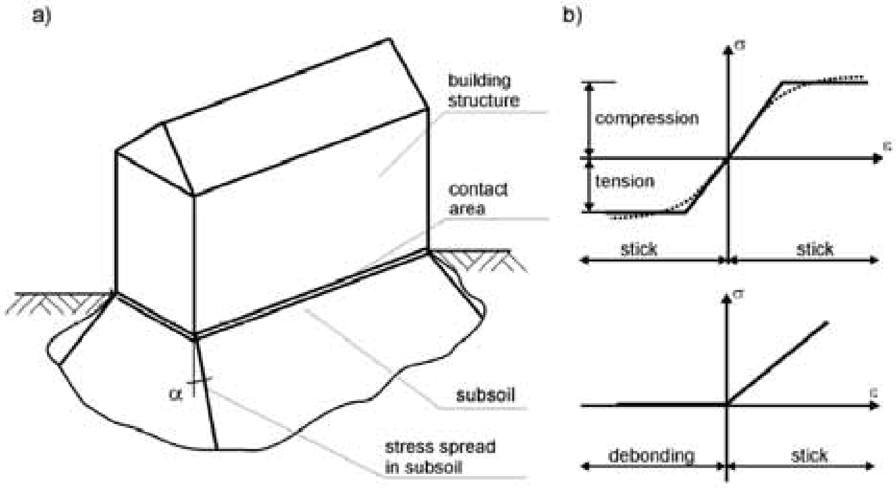


Fig. 1. a) soil-structure system, b) contact area behavior.

## Building structure model

The model is constructed using the subdivision of the structure into building elements such as wall and floor slab elements, plane and three-dimensional joints (vertical and horizontal) and lintels. Wall and floor elements, which are treated as vertical and horizontal strips, are described by deep beam scheme taking into account compression and twisting. Transverse section deformation is assumed as in Thimoshenko-type beam (Chyzy et a. 2006).

## Subsoil model

The subsoil solid elements have the shape of prism (Fig.1., Fig.2., Fig.3) and are separated from the soil space under the footings or foundations strips. Generating lines are coming from the footings edges being in contact with the soil.

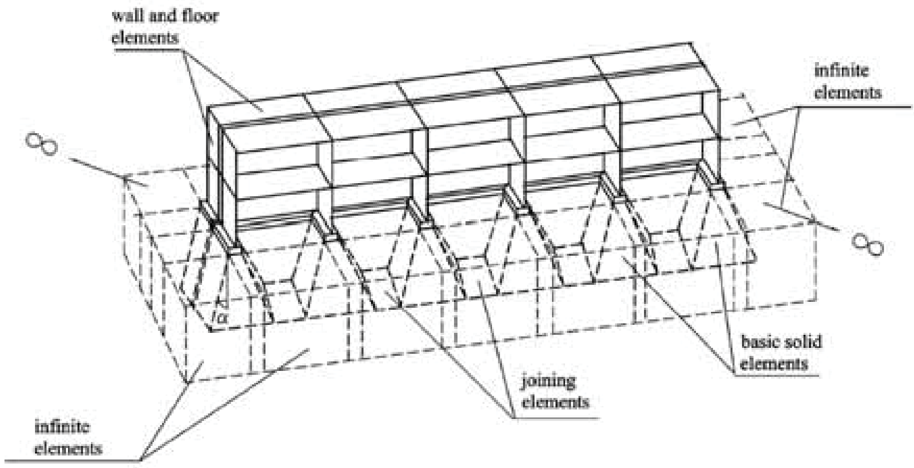


Fig. 2. View of the subsoil solids under the building.

The idea of the prism-shaped element rises from the observation, that the computation of the vertical stresses in the subsoil at any depth below a foundation is based on the assumption that the load is spread uniformly at an angle of ($90^o$ - $\alpha$) with the horizontal; but the area considered as supporting the load shall not extend beyond the intersection of ($90^o$ - $\alpha$) planes of adjacent foundations. Following the work of Andersen (1956), the angle a depends on the type of soil.

The special spatial elements are introduced to interconnect the soil solids separated under the footings. The infinite elements have been used to model far field domains (Fig.2).

The state of displacement inside the prism is assumed in three dimensional scheme as a short prismatic bar. The bar is subjected to bending and compression in the planes of building walls and is twisted and compressed as a result of the surrounding soil interaction. The displacements of horizontal sections are assumed as linear, such as in Timoshenko's beam.

The properties of the half-space determined by the soil solids and far field domains, at a given moment of time and loading, are presumed to be linear elastic.

**Interface model**

The importance of the interface behavior in the soil-structure interaction problems have been recognized for a long time. In the finite element analysis the introduction of interface ( and joint ) elements has proved to be very useful. There are various modes of deformation that an interface element can undergo: stick or no-slip, sliding or slip, separation or debonding and redebonding (Fig.1b. ) So far, various interface/joint elements have been presented. These include zero thickness interface elements reported by (Goodman et al. 1968), Day and Potts 1994), Coutinho et al. 2003) , isoparametric joint element presented by (Zienkiewicz et al 1970), simple interface elements for two and three dimensional problems described by (Beer 1985) and thin-layer element presented by (Desai et al. 1984,1988).

In the presented paper 8-node thin-layer element for interface behavior, such as that of (Desai et al. 1984) is used to describe this problem.

## The subsoil computational model

### The model formulation.

Let us consider the solid element shown in Fig.3.

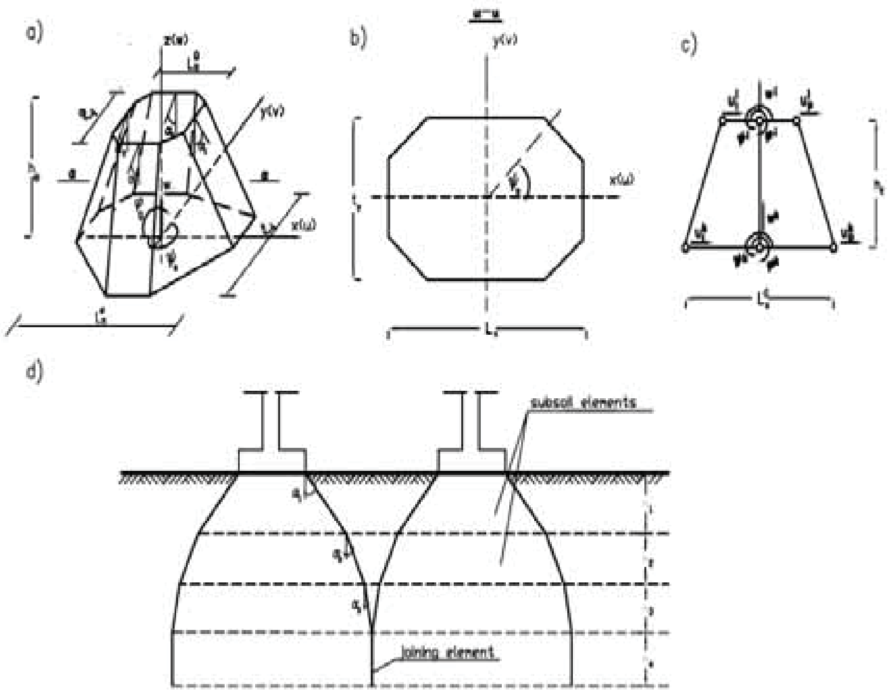

Fig. 3. The basic solid element.

The state of displacement is defined by three displacement components *u*, *v* and *w* in directions of the three coordinates *x*, *y* and *z* .

Thus

$$\mathbf{q} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u \\ 0 \\ w_0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ x\psi_s \\ -x\varphi_g \end{Bmatrix} \quad , \qquad (1)$$

where: $\phi_g$ - angle of the rotation of the cross-section in x-z plane,
$\psi_s$ - angle of the element twist in x-y plane.'

The strain field can be expressed as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xz} \\ \gamma_{zx}^{s} \end{Bmatrix} = \begin{Bmatrix} \dfrac{u_l - u_p}{L_x} \\ \dfrac{x\psi_s}{2t_y} \\ \dfrac{\partial w_0}{\partial z} - x\dfrac{\partial \varphi_g}{\partial z} \\ -\varphi_g + \dfrac{\partial (u_l + u_p)}{2\partial z} \\ 2y\dfrac{\partial \psi_s}{\partial z} \end{Bmatrix}, \tag{2}$$

where: *l*, *p* - two neighboring points between which the stresses are averaged,
$L_x$ - length of element,
$t_y$ - width of element.

The strain $\boldsymbol{\varepsilon}_z$ has been divided into $\boldsymbol{\varepsilon}_z^w$ and $\boldsymbol{\varepsilon}_z^{\varphi}$.
Hence

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z^w \\ \gamma_{xz} \\ \varepsilon_z^{\varphi} \\ \gamma_{zx}^{s} \end{Bmatrix} = \mathbf{L}\overline{\mathbf{u}} = \begin{bmatrix} \dfrac{1}{L_x} & -\dfrac{1}{L_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{x}{2t_y} \\ 0 & 0 & \dfrac{\partial}{\partial z} & 0 & 0 \\ \dfrac{\partial}{2\partial z} & \dfrac{\partial}{2\partial z} & 0 & -1 & 0 \\ 0 & 0 & 0 & -x\dfrac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 2y\dfrac{\partial}{\partial z} \end{bmatrix} \begin{Bmatrix} u_l \\ u_p \\ w_0 \\ \varphi_g \\ \psi_s \end{Bmatrix} \tag{3}$$

The stress field has a form

$$s = \mathrm{D}\, e = \mathrm{D}\,\mathrm{L}\,\bar{u} \quad , \tag{4}$$

where **D -** the constitutive relationship matrix in which the compression has been taken into account by introducing the shear coefficient *k*.

$$D = \frac{E_0}{(1+\nu)(1-2\nu)}\left(D_{\mathbf{diag}} + D_{12}\right) \quad , \tag{5}$$

$$D_{diag} = \left[1-\nu, \quad 1-\nu, \quad 1-\nu, \quad \frac{1-2\nu}{2}k, \quad 1-\nu, \quad \frac{1-2\nu}{2}\right], \tag{6}$$

$\mathbf{D}_{12}$ - six by six matrix, in which $d_{12} = d_{13} = d_{21} = d_{23} = d_{31} = d_{32} = \nu$ , and the others elements are equal 0,

ν - Poisson's ratio.

In the joined elements and in the elements which describe the far field domains the state of displacement has the form

$$\mathbf{q} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u(x,\ y,\ z) \\ v(x,\ y,\ z) \\ w(x,\ y,\ z) \end{Bmatrix} \tag{7}$$

**The Finite Element Method application**

The discrete model is formulated within the Finite Element Method in agreement with Zienkiewicz (1986). The global scheme is created by prism-shaped finite elements, joined finite elements and infinite elements.

**The basic prism-shaped finite element.**

The basic element is presented in Fig.4a.

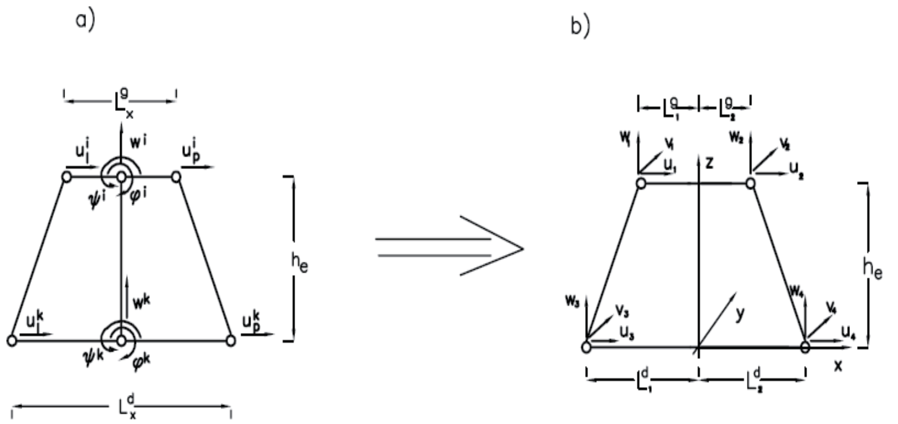


Fig. 4. The unknown displacements at corners of the element.

The displacements of an arbitrary point can be written as

$$\overline{u} = \mathbf{N}\mathbf{d}_e \tag{8}$$

where **N** - shape function matrix,

$\mathbf{d}_e$ - unknown displacements to be determined.

Components of the unknown displacement vector of the nodes “i  and “k of a basic subsoil finite element are given by

$$d_e = \{u_l^i, \quad u_p^i, \quad w_0^i, \quad j^i, \quad y^i, \quad u_l^k, \quad u_p^k, \quad w_o^k, \quad j^k, \quad y^k\}^T \tag{9}$$

The principle of virtual work has been used to establish FEM relations

$$\iiint_V (de)^T s dV - \iiint_V (d\overline{u})^T p\, dV = 0 \tag{10}$$

Hence

$$K_e d_e = P_e \tag{11}$$

where **K** - element stiffness matrix,

$$K_e = \iiint_V B^T \mathbf{D}\mathbf{B} dV \tag{12}$$

**B** - matrix of relations between strains and nodal displacements in an element,

**P** - vector of nodal forces,

$$\mathbf{B}=\begin{bmatrix} \frac{N_i}{L_x} & -\frac{N_i}{L_x} & 0 & 0 & 0 & \frac{N_k}{L_x} & -\frac{N_k}{L_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{x\ N_i}{2t_y} & 0 & 0 & 0 & 0 & \frac{x\ N_k}{2t_y} \\ 0 & 0 & \frac{\partial N_i}{\partial z} & 0 & 0 & 0 & 0 & \frac{\partial N_k}{\partial z} & 0 & 0 \\ \frac{\partial N_i}{2\partial z} & \frac{\partial N_i}{2\partial z} & 0 & -N_i & 0 & \frac{\partial N_k}{2\partial z} & \frac{\partial N_k}{2\partial z} & 0 & -N_k & 0 \\ 0 & 0 & 0 & -\frac{x\partial N_i}{\partial z} & 0 & 0 & 0 & 0 & -\frac{x\partial N_k}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 2y\frac{\partial N_i}{\partial z} & 0 & 0 & 0 & 0 & 2y\frac{\partial N_k}{\partial z} \end{bmatrix} \qquad (13)$$

$$P_e = \iiint_V N^T p\, dV \qquad (14)$$

In order to obtain more convenient computer implementation the unknown displacements are defined at corners of elements (Fig.4.).

Hence, the stiffness matrix has to be transformed according to the formula

$$\overline{K} = A^T K' A \qquad (15)$$

where **A** - transformation matrix.

$$A=\begin{bmatrix} A_{\mathbf{11}} & 0 \\ 0 & A_{\mathbf{22}} \end{bmatrix} \qquad (16)$$

$$\mathbf{A}_{11}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{L_2^g}{L_x^g} & 0 & 0 & \frac{L_1^g}{L_x^g} \\ 0 & 0 & \frac{1}{L_x^g} & 0 & 0 & -\frac{1}{L_x^g} \\ 0 & -\frac{1}{L_1^g} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{L_2^g} & 0 \end{bmatrix} \qquad \mathbf{A}_{22}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{L_2^d}{L_x^d} & 0 & 0 & \frac{L_1^d}{L_x^d} \\ 0 & 0 & \frac{1}{L_x^d} & 0 & 0 & -\frac{1}{L_x^d} \\ 0 & -\frac{1}{L_1^d} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{L_2^d} & 0 \end{bmatrix} \qquad (17)$$

The new vector of unknown has the form

$$d_e = \{u_1,\ v_1,\ w_1,\ u_2,\ v_2,\ w_2,\ u_3,\ v_3,\ w_3,\ u_4,\ v_4,\ w_4\}^T \qquad (18)$$

**Joining finite elements and infinite elements.**

Joining finite elements which are used in presenting subsoil model are shown in Fig.5. The unknown displacements of the discrete model are the same as in the case of the three dimensional finite elements.

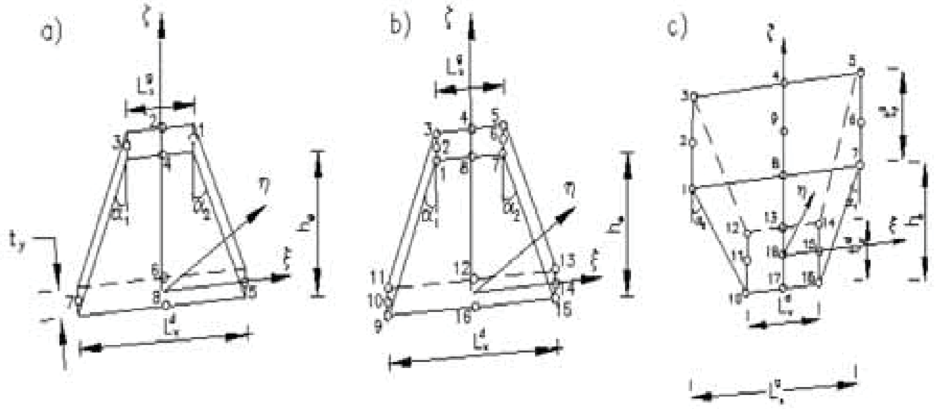

Fig.5. Joining elements.

The stiffness matrices of those elements are created by using classical finite elements (8-node, 16-node and 18-node).

Infinite elements are used to model far field domains. These help in reducing the total number of elements and model the domain at infinity in a better way.

Various procedures exist to model the unbounded region in the far field. These include Bettes (1984,1992) who reported 2D and 3D mapped element with expotential decay, where a finite element is stretched to an infinite element, (Beer & Meek 1981) presented how the infinite boundary can be accommodated within the finite element analysis by developing special elements which extend to infinity in one direction and (Viladkar et al. 1990) presented some new 3D mapped infinite elements with (1/r) and ($1/\sqrt{r}$) type of decay.

The infinite elements used in the model are presented in Fig.6. The displacement vector is defined by three displacement components *u*, *v* and *w*. In agreement with (Bettes (1984, 1992) the infinite elements stiffness matrices are based on the finite element method, but the element shape functions have to be modified.

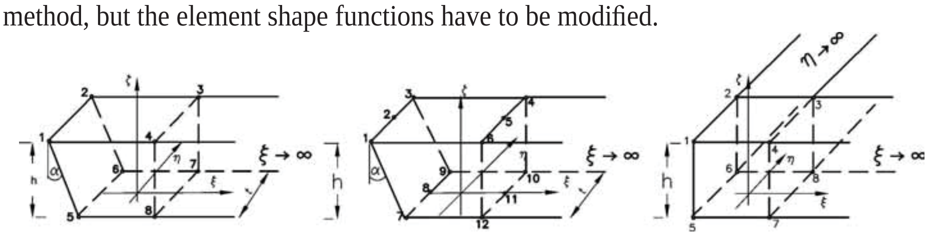

Fig. 6. Infinite elements.

In this case the shape function , such as that of Bettes (1992) formulation, has been used.

$$N_i(\xi,\eta,\varsigma) = f_i(\xi,\eta,\varsigma)M_i(\xi,\eta,\varsigma) \tag{19}$$

where $N_i$ - infinite element shape function,
$M_i$ - finite element shape function,
$f_i$ - decay function.

The decay functions used for the elements presented in Fig.6. have the form

$$f_i(\xi,\eta,\varsigma,) = \exp\left(\frac{\xi_i-\xi}{L}\right) \qquad \xi \to \infty \ ,$$
$$f_i(\xi,\eta,\varsigma) = \exp\left(\frac{\xi_i+\eta_i-\xi-\eta}{L}\right) \qquad \xi \to \infty\ ,\ \eta \to \infty \ , \tag{20}$$
$$f_i(\xi,\eta,\varsigma,) = \exp\left(\frac{\xi-\xi_i}{L}\right) \qquad \xi \to -\infty \ .$$

where
L - the severity of the decay described by Bettes (1992).

**Soil - structure interaction model.**

The soil - structure interaction FEM model has the form

$$\begin{bmatrix} K_{kk} & K_{kkt} & 0 & 0 \\ K_{ktk} & K_{ktkt} & K_{ktg} & 0 \\ 0 & K_{gkt} & K_{gg} & K_{gn} \\ 0 & 0 & K_{ng} & K_{nn} \end{bmatrix} \begin{bmatrix} u_k \\ u_{kt} \\ u_g \\ u_n \end{bmatrix} = \begin{bmatrix} P_k \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

where k - degrees of freedom of the structure,
g - degrees of freedom of the subsoil,
n - degrees of freedom of the surrounding ground,
kt - degrees of freedom of the interface elements
**K** - submatrices,
**u** - displacement vector,
**P** - nodal forces vector.

In the presented paper 8-node thin-layer element (Fig.7) is used to describe the interface behavior in the soil-structure interaction problems

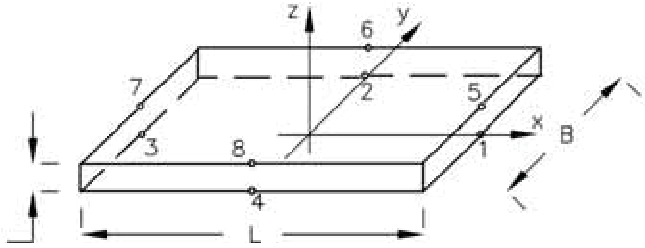


Fig. 7. The interface element.

The interface element is assumed to have a finite thickness t. The value of t can be chosen such that the ratio t/B is in the range of 0.001-0.100 in which B – width of adjoining solid element. For further details of this interface element see Desai et al. (1984).

## Numerical examples

Several numerical tests have been solved to verify the correctness, accuracy and efficiency of the presented subsoil model.

### Numerical test No.1.

The horizontal displacements *u*, caused by force P=10kN, have been determined in the system presented in Fig.8. The system consists of four basic subsoil elements and joining element. The verification of displacements convergence of the proposed subsoil model by the comparison with the results obtained by using commercial software MSC/NASTRAN have been done. Material and geometrical data: $E_0$ = 80 MPa, ν = 0.3, t = 1 m, L=4.0 m, h = 6 m. Table 1 shows the horizontal displacements *u* (Fig.8). Fig.9. shows the displacements values determined in point 2 in comparison to the height of elements and number of nodes.

Error has been expressed by relative Euclidean norm of displacements obtained by proposed model and comparative method. The obtained results show that error value is equal 3.5% but the number of unknowns has been significantly reduced (10.5 times).

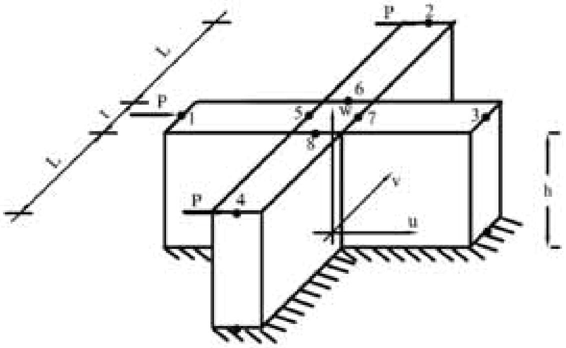


Fig. 8. Tested system.

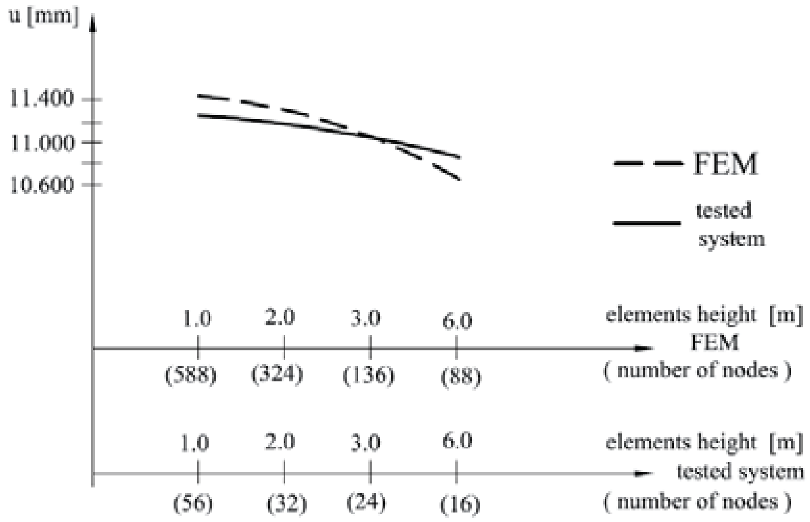


Fig. 9 Displacements values determined in point 2 in comparison to the height of elements and number of nodes.

Tab. 1. The horizontal displacements *u* (Fig.8)

| method \ displacement | | u [mm] | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 2.0x1.0x1.0 | 1.689 | 11.456 | 0.842 | 11.456 | 0.952 | 0.906 | 0.892 | 0.906 |
| FEM | 2.0x1.0x2.0 | 1.580 | 11.320 | 0.831 | 11.320 | 0.945 | 0.888 | 0.883 | 0.888 |
| 20-node | 4.0x1.0x3.0 | 1.476 | 11.070 | 0.825 | 11.070 | 0.939 | 0.880 | 0.871 | 0.880 |
| | 4.0x1.0x6.0 | 1.312 | 10.631 | 0.813 | 10.631 | 0.923 | 0.876 | 0.858 | 0.876 |
| | $h_i$=1.0m | 1.521 | 11.234 | 0.832 | 11.234 | 0.946 | 0.891 | 0.883 | 0.891 |
| proposed | $h_i$=2.0m | 1.496 | 11.157 | 0.828 | 11.157 | 0.943 | 0.886 | 0.879 | 0.886 |
| elements | $h_i$=3.0m | 1.483 | 11.002 | 0.826 | 11.002 | 0.941 | 0.884 | 0.874 | 0.884 |
| | $h_i$=6.0m | 1.392 | 10.826 | 0.822 | 10.826 | 0.934 | 0.881 | 0.861 | 0.881 |

The test proved that proposed elements give good accuracy of the results with significantly reduced number of unknowns.

**Numerical test No.2.**

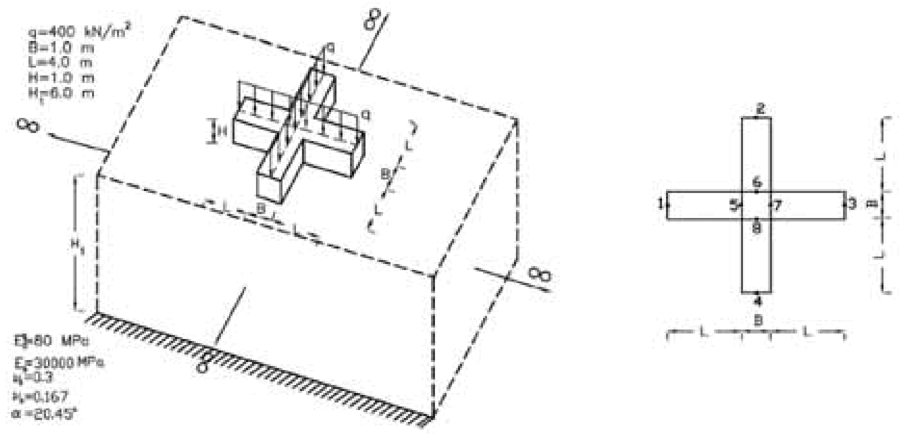


Fig. 10. Analyzed foundations system.

For the foundations system presented in Fig.10. the settlement analysis has been done. The results have been compared with the settlements received by using classic finite elements and commercial software MSC/NASTRAN. Material and geometrical data are shown in Fig.10.

Using proposed elements the subsoil has been divided into 6 layers of 1.0 m thickness.

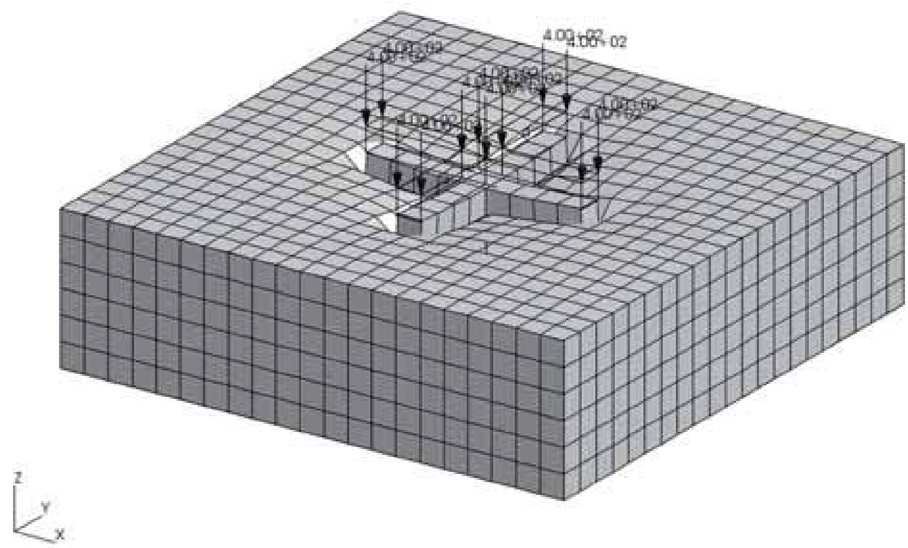

Fig. 11. The discretization of the system by using MSC/NASTRAN and the subsoil deformation caused by existing load.

Tab. 2. The settlement results (Fig. 10).

| Points, where settlements has been determined | Proposed model [mm] | MSC/NASTRAN [mm] |
|---|---|---|
| 1 | 5.56 | 5.40 |
| 2 | 5.56 | 5.40 |
| 3 | 5.56 | 5.40 |
| 4 | 5.56 | 5.40 |
| 5 | 6.63 | 6.24 |
| 6 | 6.63 | 6.24 |
| 7 | 6.63 | 6.24 |
| 8 | 6.63 | 6.24 |

The foundations have been described 4 finite elements proposed by Chyzy et al. (2006). The subsoil and foundations system gives 532 nodes. In classic FEM analysis (by using MSC/NASTRAN ) the three dimensional 8-node elements $1.0\times1.0\times1.0\text{m}$ have been used to discretize the subsoil of $21.0\times21.0\times6.0\text{m}$ dimensions and the foundations. This system gives 3432 nodes. Fig.11 shows the discretization of the system by using MSC/NASTRAN and the subsoil deformation caused by existing load. Table 2 shows the settlement results.

Error has been expressed by relative Euclidean norm of settlements obtained by proposed model and comparative method. The obtained results show that error value is equal 5%.

## Summary and conclusion

The semi-analytic description of subsoil for static analysis building structures loads has been presented. The model, which is formulated on the basis of the finite element method, takes into account the complicated characteristic of subsoil strata and allows the description to be made in three - dimensional scheme. The method formulation allows in a quickly way to obtain results convergence. The accuracy of the described model depends (like in classic FEM) on the division of the subsoil into constituent elements. Good accuracy (about 5%) has been obtained, what is acceptable in engineering practice. The infinite elements, which are used to model far field domain, help in reducing the total number of elements. The number of unknowns is small in comparison with the number of unknowns used for structure and subsoil description in classic FEM (from several to a dozens times less for the real structures) so the analysis of large building structures in three-dimensional scheme taking into account soil-structure interaction can be carried out easily in relative short time on commonly used hardware of PC class (the computational time is from dozens to several-hundreds times less in comparison with the commercial software).

**References:**


1. Andersen P. (1956), *Substructure analysis and design*, The Ronald Press Company.
2. Beer G., Meek J. L. (1981), *Infinite domain elements*, International Journal for Numerical Methods in Engineering, 17, 43-52.
3. Beer G. (1985), *An isoparametric joint/interface element for finite element analysis* International Journal for Numerical Methods in Engineering, 21, 293-300.
4. Bettes P. (1992), Infinite elements, Penshaw Press.
5. Bettess P., Bettess J.A. (1984), *Infinite elements for static problems* , Engineering Computations, 1, 4-16.
6. Choi C. K., Cheung K.Y. (1996), *Three-dimensional non-conforming 8-node finite elements with rotational degrees of freedom*, Structural Engineering and Mechanics, 4(5), 41-61.
7. Choi C. K., Kim S.H., Park Y. M., Cheun K.Y. (1998), *Two-dimensional nonconforming finite elements: A state-of-art*. Structural Engineering and Mechanics, 6(1), 41-61.
8. Choi C. K., Lee P. S., Park Y. M (1999), *Defect-free 4-node flat shell element: NMS-4F element*, Structural Engineering and Mechanics, 8(2), 207-231.
9. Chyzy T., Kretowska J., Miedzialowski Cz. (2006), *Analysis of 3D building structures dynamic response* , Structural Engineering and Mechanics, 22(1),
10. Coutinho A.L.G.A., Martins M.A.D., and Sydenstricker R.M. (2003), *Simple zero thicknes kinematically consistent interface elements* , Computers and Geotechnics, 30, 347-374.
11. Day R. A., Potts D.M. (1994), *Zero thickness interface elements - numerical stability and application* , International Journal for Numerical and Analytical Methods in Geomechanics, 18, 689-708.
12. Desai C. S., Zamann M. M., Lightner J. G., Siriwardane H.J. (1984), *Thin-layer elements for interfaces and joints* , International Journal for Numerical and Analytical Methods in Geomechanics, 8(11), 19-43.
13. Desai C.S., Nagaraj B.K. (1988), *Modeling for cyclic normal and shear behavior of interfaces* , Journal of Engineering Mechanics, 114, 1198-217.
14. Goodman R.E., Taylor R.L., Brekkee T.L. (1968), *A model for the mechanics of joined rock* , J Soil Mech Foundations Div, ASCE, 94(3), 637-659.
15. Kundu T., Mathur R.P., Desai C.S. (1991), *Three-dimensional soil-structure interaction analysis: deformable structures in multilayered soil mass* , Engineering Computations, 8, 153-180.
16. Miedzialowski Cz. (1995), *Three dimensional modelling wall structures for buildings , Archives of Civil Engineering*, XLI(2), 195-212.
17. Miedzialowski Cz. (1996), *Modeling of subsoil flexibility in computations of complex three-dimensional structural schemes of buildings* , Archives of Civil Engineering, XLII(1), 83-102.
18. Viladkar M.N., Godbole P.N., Noorzaei J. (1990), *Some new 3-D infinite element*, Computers & Structures, 34, 455-467.

19. Viladkar M.N., Godbole P.N., Noorzaei J. (1995), *Elasto-plastic analysis for soil-structure interaction in framed structures* Computers & Structures, 55(5), 797-807.
20. Wang Y.H., Cheung Y.K. (2001), *Plate on cross-anisotropic foundation analyzed by the finite element method*, Computers and Geotechnics, 28, 37-54.
21. Zienkiewicz O.C., Best B., Dullage C. (1970), *Analysis of nonlinear problem in rock mechanics with particular reference to joined rock systems* .. Proc. 2nd Congress of the International Society for rock mechanics, 501-509.
22. Zienkiewicz O. C. (1986). *The finite element method*, McGraw-Hill .